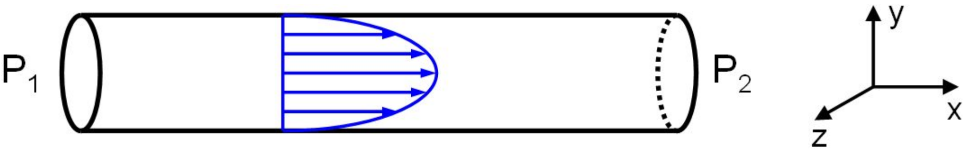
EXAMPLE: Water Flow in a Pipe



$P_1 > P_2$ Velocity profile is parabolic (we will learn why it is parabolic later, but since friction comes from walls the shape is intuitive)

The pressure drops linearly along the pipe.

Does the water slow down as it flows from one end to the other?

Only component of velocity is in the x-direction.

$$\vec{v} = v_x \vec{i}$$

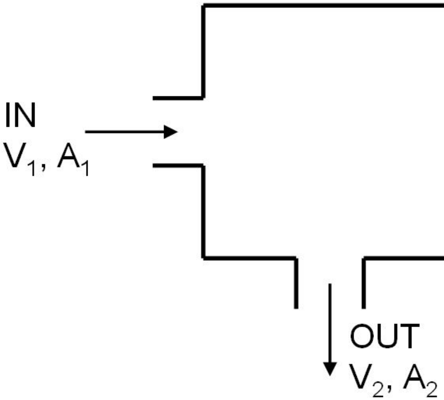
$$v_y = v_z = 0$$

Incompressible Continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$\therefore \quad \frac{\partial v_x}{\partial x} = 0$ and the water does not slow down.

EXAMPLE: Flow Through a Tank



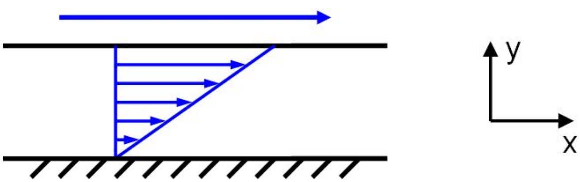
V = constant (always full)

Integral Mass Balance: $\int_S (\vec{v} \cdot \vec{n}) dA = 0$

$v_1 A_1 = v_2 A_2 \equiv Q$

Constant *volumetric flow rate* Q.

EXAMPLE: Simple Shear Flow



$$v_y = v_z = 0 \quad v_x = v_x(y)$$

$$\vec{\nabla} \cdot v \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

satisfied identically

NAVIER-STOKES EQUATIONS (p. 1)

(in the limit of slow flows with high viscosity)

$$\text{Reynolds Number:} \qquad R_e \equiv \frac{\rho v D}{\eta} \tag{1-62}$$

$\rho =$ density
$\eta =$ viscosity
$v =$ typical velocity scale
$D =$ typical length scale

For $R_e \ll 1$ have *laminar flow* (no turbulence)

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P + \rho \vec{g} + \eta \nabla^2 \vec{v}$$

Vector equation (thus really three equations)

The full Navier-Stokes equations have other nasty inertial terms that are important for low viscosity, high speed flows that have turbulence (airplane wing).

NAVIER-STOKES EQUATIONS (p. 2)

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}P + \rho\vec{g} + \eta\nabla^2\vec{v}$$

$$\frac{\partial\vec{v}}{\partial t} = \text{acceleration}$$

$$\rho = \frac{\text{mass}}{\text{unit volume}}$$

$$\rho\frac{\partial\vec{v}}{\partial t} = \frac{\text{force}}{\text{unit volume}} \quad (\vec{F} = m\vec{a}) \quad \text{Newton's } 2^{nd} \text{ Law}$$

Navier-Stokes equations are a *force balance* per unit volume

What accelerates the fluid?

$-\vec{\nabla}P =$ Pressure Gradient

$\rho\vec{g} =$ Gravity

$\eta\nabla^2\vec{v} =$ Flow (fluid accelerates in direction of increasing velocity gradient. Increasing $\nabla\vec{v} \quad \Rightarrow \quad \nabla^2\vec{v} > 0$

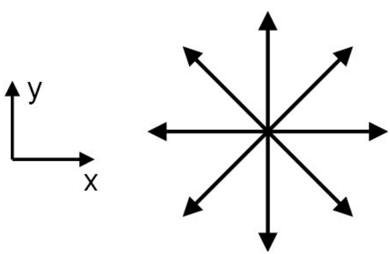
GENERAL FLUID MECHANICS SOLUTIONS

Navier-Stokes equations + Continuity + Boundary Conditions

Four coupled differential equations!

Always look for ways to simplify the problem!

EXAMPLE: 2D Source Flow Injection Molding a Plate



1. Independent of time
2. 2-D $\Rightarrow v_z = 0$
3. Symmetry $\Rightarrow$ Polar Coordinates
4. Symmetry $\Rightarrow v_\theta = 0$

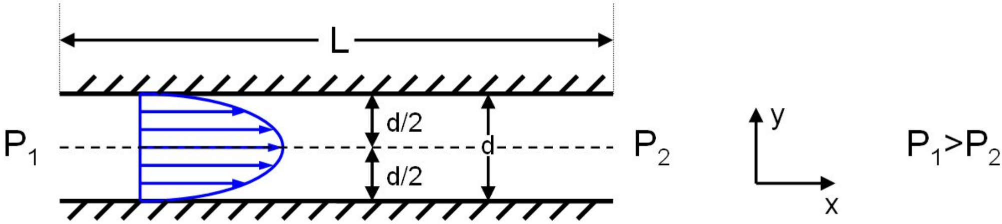
Continuity equation $\quad \vec{\nabla} \cdot \vec{v} = \frac{1}{r}\frac{d}{dr}(r v_r) = 0$

$$r v_r = \text{constant}$$

$$v_r = \frac{\text{constant}}{r}$$

Already know the way velocity varies with position, and have not used the Navier-Stokes equations!

EXAMPLE: Poiseuille Flow between Parallel Plates (important for injection molding) (P. 1)



Independent of time

$v_y = v_z = 0$

Cartesian coordinates

Continuity:

$$\frac{\partial v_x}{\partial x} = 0 \quad v_x = v_x(y)$$

Navier-Stokes equation:

$$-\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} = 0 \quad \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$$

$P = P(x) \quad v_x = v_x(y)$

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

How can $f(x) = h(y)$? Each must be constant!
$\frac{\partial P}{\partial x} = C_1 \quad P = C_1 x + C_2$

B.C. $x = 0 \quad P = P_1 \Rightarrow C_2 = P_1$
$x = L \quad P = P_2 \Rightarrow C_1 = -\Delta P/L$ where : $\Delta P \equiv P_1 - P_2$
$P = P_1 - \frac{\Delta P x}{L}$

EXAMPLE: Poiseuille Flow between Parallel Plates (important for injection molding) (P. 2)

$$\mu \frac{\partial^2 v_x}{\partial y^2} = C_1 = -\Delta P/L$$

$$\frac{\partial^2 v_x}{\partial y^2} = -\frac{\Delta P}{\mu L}$$

$$\frac{\partial v_x}{\partial y} = -\frac{\Delta P}{\mu L} y + C_3$$

$$v_x = -\frac{\Delta P}{2\mu L} y^2 + C_3 y + C_4$$

B.C. NO SLIP — top plate $y = d/2 \quad v_x = 0$

bottom plate $y = -d/2 \quad v_x = 0$

$$0 = \frac{-\Delta P}{8\mu Ł} d^2 + C_3 \frac{d}{2} + C_4$$

$$0 = \frac{-\Delta P}{8\mu Ł} d^2 - C_3 \frac{d}{2} + C_4$$

$$\therefore C_3 = 0 \qquad C_4 = \frac{\Delta P d^2}{8\mu L}$$

$$v_x = \frac{\Delta P}{2\mu l} \left[\frac{d^2}{4} - y^2 \right]$$ Parabolic velocity profile

EXAMPLE: Poiseuille Flow between Parallel Plates (important for injection molding) (P. 3)

Where is the velocity largest?
Maximum at $\frac{\partial v_x}{\partial y} = 0 = -\frac{\Delta P}{\mu L} y$
maximum at $y = 0$ centerline

What is the average velocity?

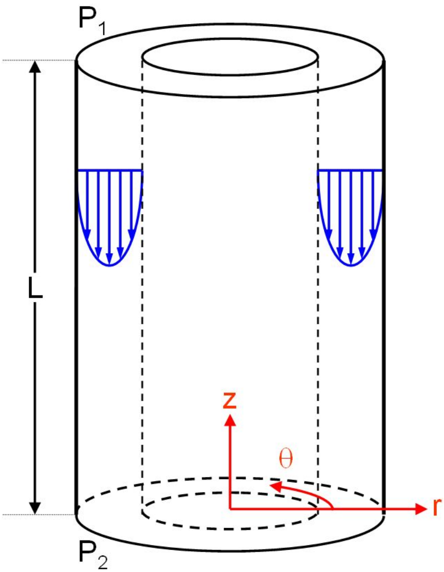
$$v_{ave} = \frac{\int_A v_x dA}{\int_A dA} = \frac{1}{A} \int_A v_x dA \qquad A = zd$$

$$v_{ave} = \frac{1}{zd} \int_0^z \int_{-d/2}^{d/2} v_x dy dz = \frac{1}{d} \int_{-d/2}^{d/2} \frac{\Delta P}{2\mu L} \left[\frac{d^2}{4} - y^2 \right] dy$$

$$v_{ave} = \frac{\Delta P}{2\mu L d} \left[\frac{d^2}{4} y - \frac{y^3}{3} \right]_{-d/2}^{d/2} = \frac{\Delta P d^2}{12 \mu L}$$

For constant ΔP, μ, L: double $d \Rightarrow$ quadruple v

EXAMPLE: Poiseuille Flow in an Annular Die (important for blow molding) (P. 1)



$P_1 > P_2$

Independent of Time

Cylindrical Coordinates

$v_r = v_\theta = 0$

$v_z = v_z(r)$

Continuity: $\frac{\partial v_z}{\partial z} = 0$

Navier-Stokes equation:

$$\frac{\partial P}{\partial z} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

$f(z) = g(r) =$ a constant

Split into two parts - Pressure Part:

$\frac{\partial P}{\partial z} = C_1 \qquad P = C_1 z + C_2$

B.C. $\quad z = 0 \quad P = P_2 \Rightarrow C_2 = P_2$

$z = L \quad P = P_1 \Rightarrow C_1 = \Delta P / L \qquad$ where : $\Delta P \equiv P_1 - P_2$

$P = P_2 + \frac{\Delta P}{L} z$

$P = P_2 + \frac{\Delta P}{L} z$ analogous to Poiseuille flow between parallel plates.

EXAMPLE: Poiseuille Flow in an Annular Die (important for blow molding) (P. 2)

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]=\frac{\Delta P}{L}$$

$$r\frac{\partial v_z}{\partial r}=\frac{\Delta P}{2\mu L}r^2+C_3$$

$$\frac{\partial v_z}{\partial r}=\frac{\Delta P}{2\mu L}r+\frac{C_3}{r}$$

$$v_z=\frac{\Delta P}{4\mu L}r^2+C_3\ln r+C_4$$

B.C. NO SLIP at $r=R_i,\quad v_z=0$
at $r=R_0,\quad v_z=0$

$$0=\frac{\Delta P}{4\mu L}R_i^2+C_3\ln R_i+C_4$$

$$0=\frac{\Delta P}{4\mu L}R_0^2+C_3\ln R_0+C_4$$

subtract $\quad 0=\frac{\Delta P}{4\mu L}(R_0^2-R_i^2)+C_3\ln\left(\frac{R_0}{R_i}\right)$

$$C_3=-\frac{\Delta P(R_0^2-R_i^2)}{4\mu L\ln(R_0/R_i)}$$

$$C_4=-\frac{\Delta P}{4\mu L}\left[R_0^2-\frac{(R_0^2-R_i^2)\ln R_0}{\ln(R_0/R_i)}\right]$$

EXAMPLE: Poiseuille Flow in an Annular Die (important for blow molding) (P. 3)

$$v_z = \frac{\Delta P}{4\mu L}\left[r^2 - \frac{(R_0^2 - R_i^2)}{\ln(R_0/R_i)}\ln r - R_0^2 + \frac{(R_0^2 - R_i^2)}{\ln(R_0/R_i)}\right]$$

$$v_z = \frac{\Delta P R_0^2}{4\mu L}\left[-1 + \frac{r^2}{R_0^2} - \frac{(R_0^2 - R_i^2)}{\ln(R_0/R_i)}\ln(r/R_0)\right]$$

$r < R_0$ always, so $v_z < 0$

Leading term is parabolic in r (like the flow between plates) but this one has a logarithmic correction.

What is the volumetric flow rate?

$$Q = \int_A v_z dA = \int_{R_i}^{R_0} v_z 2\pi r dr$$

$$Q = \frac{\pi \Delta P R_0^4}{8\mu L}\left[-1 + \left(\frac{R_i}{R_0}\right)^4 + \frac{(1 - (R_i/R_0)^2)^2}{\ln(R_0/R_i)}\right]$$

GENERAL FEATURES OF NEWTONIAN POISEUILLE FLOW

Parallel Plates: $$Q = \frac{\Delta P d^3 W}{12 \mu L}$$

Circular Tube: $$Q = \frac{\pi \Delta P R^4}{8 \mu L}$$

Annular Tube: $$Q = \frac{\pi \Delta P R_0^4}{8 \mu L} f(R_i/R_0)$$

Rectangular Tube: $$Q = \frac{\Delta P d^3 w}{12 \mu L}$$

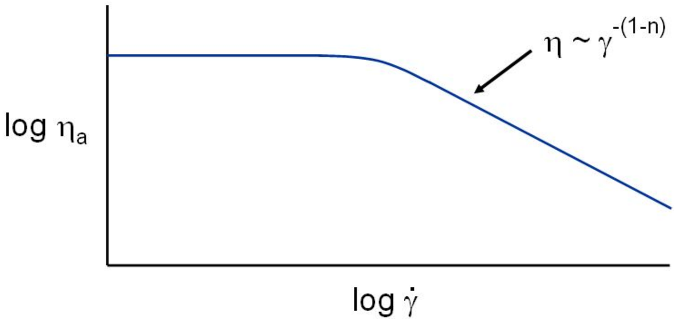
All have the same general form:

$$\left.\begin{array}{l} Q \sim \Delta P \\ Q \sim 1/\mu \\ Q \sim 1/L \end{array}\right\} \text{Weak effects of pressure, viscosity and flow length}$$

$Q \sim R^4$ or $d^3 w$ Strong effect of size.

In designing and injection mold, we can change the runner sizes.

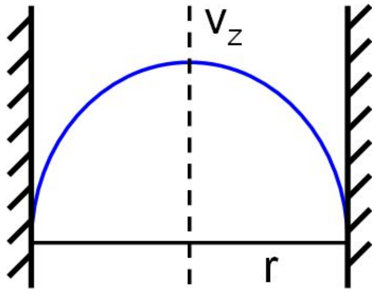
NON-NEWTONIAN EFFECTS



EXAMPLE: Poiseuille Flow in a Circular Pipe

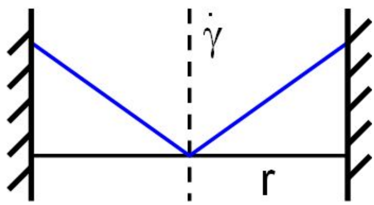
Newtonian Velocity Profile:

$$v_z = \frac{\Delta P R^2}{4\mu L}\left[1 - (r/R)^2\right]$$

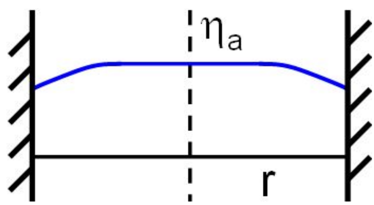


Shear Rate:

$$\dot{\gamma} = -\frac{\partial v_z}{\partial r} = \frac{\Delta P r}{2\mu L}$$



Apparent Viscosity: Viscosity is lower where $\dot{\gamma}$ is higher



Real Velocity Profile:
Lower η_a increases v_z
non-parabolic